Višja raven, splošna matura, 9.junij 2018

Naloga 1 je obvezna.

1. Dana je funkcija **f** za katero velja da je $f(2x-1)=\frac{x-1}{x^2+x}$ za poljuben $x\in\mathbb{R}\setminus\{0,-1\}$.

1.1.Dokažite da funkcijo **f** lahko podamo s predpisom $f(x)=\frac{2x-2}{x^2+4x+3}$.

1.2.Izračunajte, v katerih vrednostih spremenljivke **x** ima funkcija **f** stacionarne točke.

1.3.Določite vse ničle in pole funkcije **f** in zapišite enačbo njegove vodoravne asimptote.

1.4.Narišite krivulji dani z enačbami $y=|f(x)|$ in $y=f(|x|)$.

Rešitve:

1.1. $$f(x) = \frac{2x-2}{x^2+4x+3}$$

$$f(2x-1) = \frac{2(2x-1)-2}{(2x-1)^2+4(2x-1)+3} =$$

$$= \frac{4x-2-2}{4x^2-4x+1+8x-4+3} = \frac{4x-4}{4x^2+4x} =$$

$$= \frac{4(x-1)}{4(x^2+x)} = \frac{x-1}{x^2+x}$$

1.2. $$f'(x) = \frac{(2x-2)'\cdot(x^2+4x+3)-(2x-2)(x^2+4x+3)'}{(x^2+4x+3)^2}$$

$$f'(x) = \frac{(2)(x^2+4x+3)-(2x-2)\cdot(2x+4)}{(x^2+4x+3)^2} =$$

$$= \frac{2x^2+8x+6-4x^2-8x+4x+8}{(x^2+4x+3)^2} =$$

$$= \frac{-2x^2+4x+14}{(x^2+4x+3)^2}$$

$f'(x) = 0$ $\qquad -2x^2+4x+14=0 \;/:(-2)$

$$x^2-2x-7=0$$

$$D = b^2-4ac = (-2)^2-4\cdot 1\cdot(-7) =$$

$$= 4+28 = 32$$

$\sqrt{D} = \sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$

$$x_{1/2} = \frac{-(-2) \pm 4\sqrt{2}}{2} = \frac{2 \pm 4\sqrt{2}}{2}$$

$x_1 = 1 + 2\sqrt{2} = 3{,}828$

$x_2 = 1 - 2\sqrt{2} = -1{,}828$

$$y_1 = f(1+2\sqrt{2}) = \frac{2(1+2\sqrt{2})-2}{(1+2\sqrt{2})^2+4(1+2\sqrt{2})+3} = 0{,}172$$

$S_1(3{,}83;\ 0{,}17)$ $\qquad$ $S_2(-1{,}83;\ 5{,}83)$

$$y_2 = f(1-2\sqrt{2}) = \frac{2(1-2\sqrt{2})-2}{(1-2\sqrt{2})^2+4(1-2\sqrt{2})+3} = 5{,}828$$

1.3. $2x - 2 = 0$

Ničla $x = 1$ I

$y = 0$

Vodoravna asimptota

$x^2 + 4x + 3 = 0$

$(x+3)(x+1) = 0$

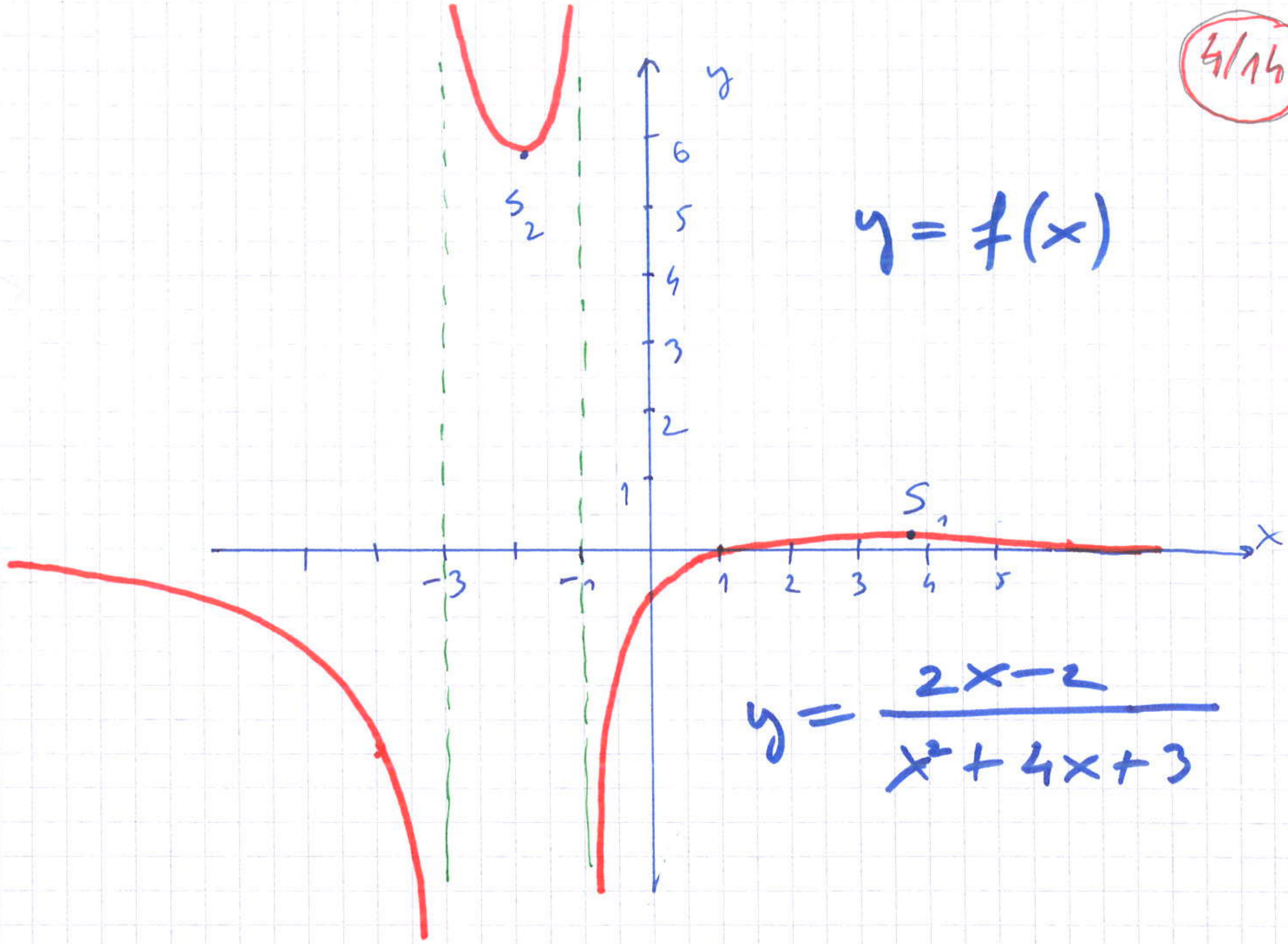
$x_1 = -3$ I

$x_2 = -1$ I

Pola

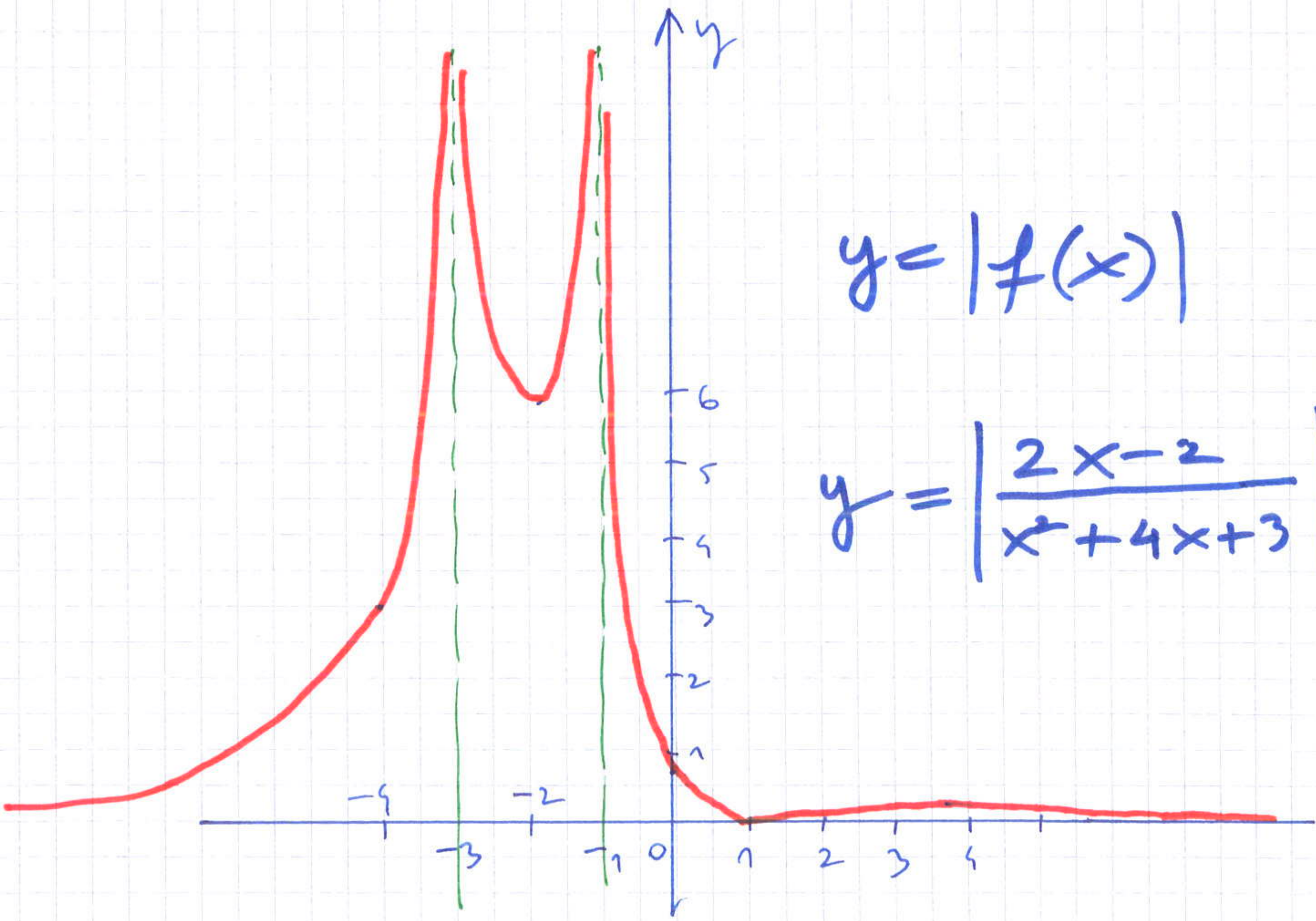
$$y = f(x)$$

$$y = \frac{2x-2}{x^2+4x+3}$$

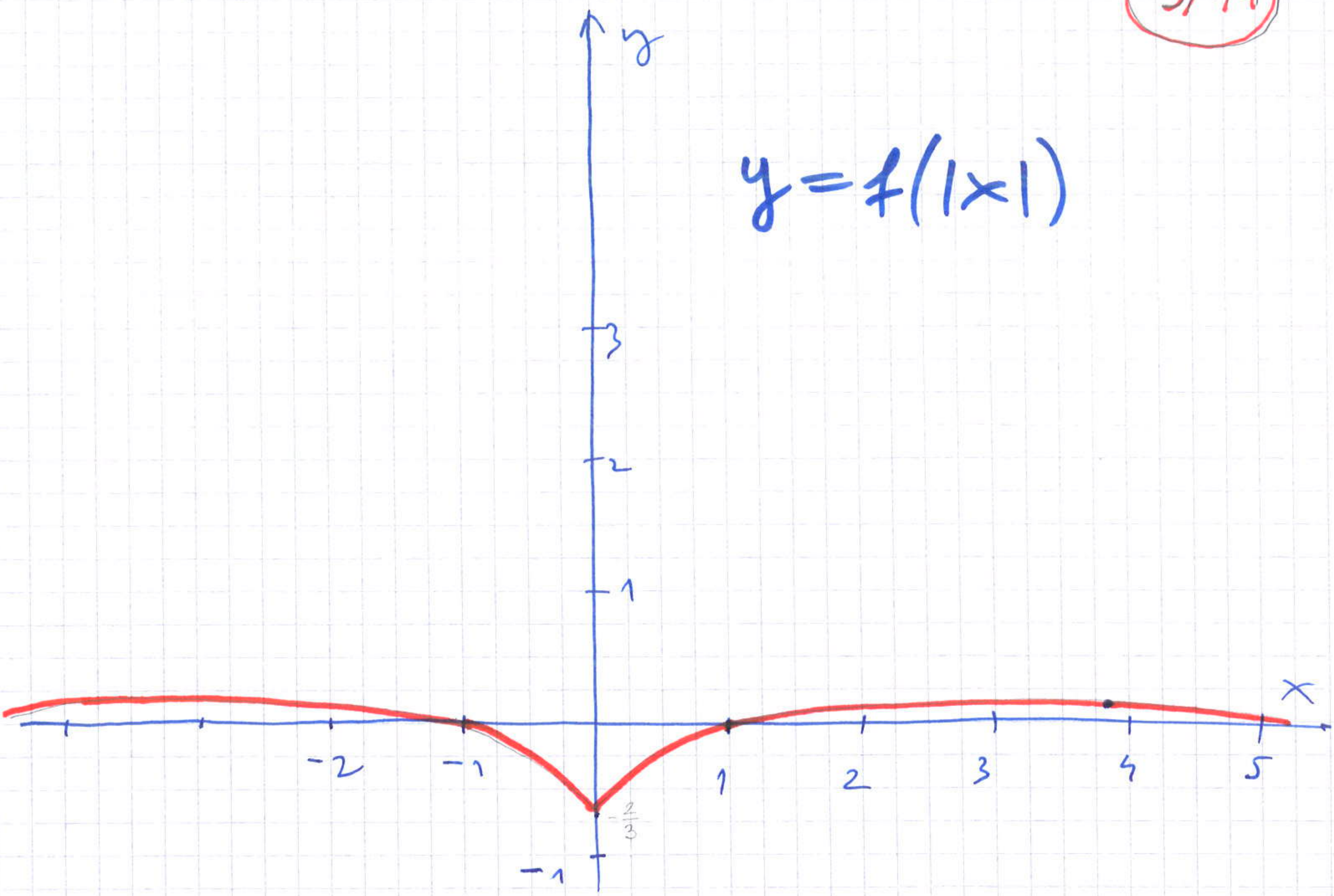


$$y = |f(x)|$$

$$y = \left| \frac{2x-2}{x^2+4x+3} \right|$$



$$y = f(|x|)$$



$$y = \frac{2|x| - 2}{|x|^2 + 4|x| + 3}$$

Naloga 2 je obvezna.

2.Rešite naloge o množicah točk v ravnini.

2.1.V koordinatnem sistemu ponazorite množico točk:

$$\mathbf{A} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}, (|\boldsymbol{x} - \mathbf{3}| < \mathbf{2}) \wedge (\boldsymbol{y} > -\mathbf{1}) \wedge (\boldsymbol{x} + \boldsymbol{y} < \mathbf{6})\}\ .$$

2.2. Izračunajte ploščino območja:

$$\mathbf{B} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}, (|\boldsymbol{x} - \mathbf{3}| \leq \mathbf{2}) \wedge (\boldsymbol{y} \geq -\mathbf{1}) \wedge (\boldsymbol{x} + \boldsymbol{y} \leq \mathbf{2018})\}\ .$$

2.3. V koordinatnem sistemu ponazorite množici točk:

$$C_1 = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}, (\boldsymbol{x} - \mathbf{1})^2 + \boldsymbol{y}^2 = \mathbf{2}^{-2}\}$$

$$C_2 = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}, (\boldsymbol{x} - \mathbf{2})^2 + \boldsymbol{y}^2 = \mathbf{2}^{-3}\}$$

2.4. Za vsako naravno število $\mathbf{n}$ je

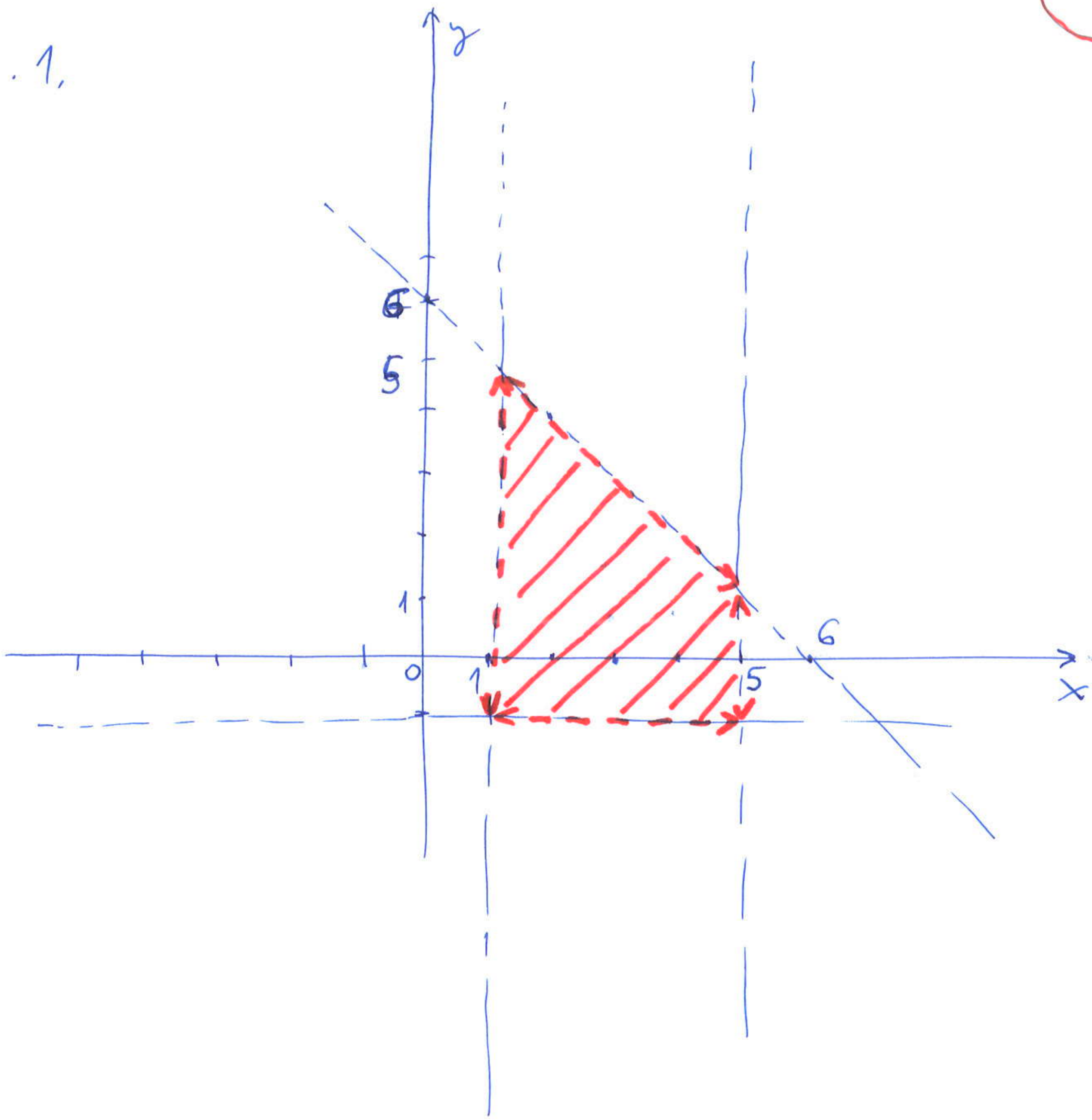
$$D_n = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}, (\boldsymbol{x} - \boldsymbol{n})^2 + \boldsymbol{y}^2 = \mathbf{2}^{-\boldsymbol{n}-\mathbf{1}}\}$$

Izračunajte ploščino območja:

$$D_1 \cup D_2 \cup D_3 \cup \ldots \cup D_n \cup \ldots$$

Rešitve:

2.1.



2.2.

$$S_{trapez} = \frac{1}{2}(a+c)\cdot v$$

$x + y = 2018$

$x = 1$

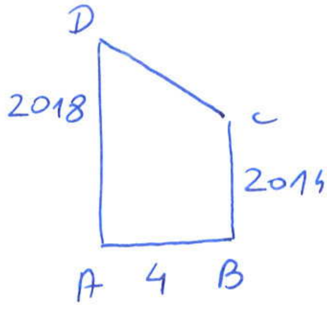
$y = 2018 - 1$

$y_D = 2017$

$x = 5$

$y = 2018 - 5$

$y_C = 2013$



$$S = \frac{1}{2}(2018 + 2014)\cdot 4 = 8064$$

2.3. $(x-p)^2+(y-q)^2=r^2$

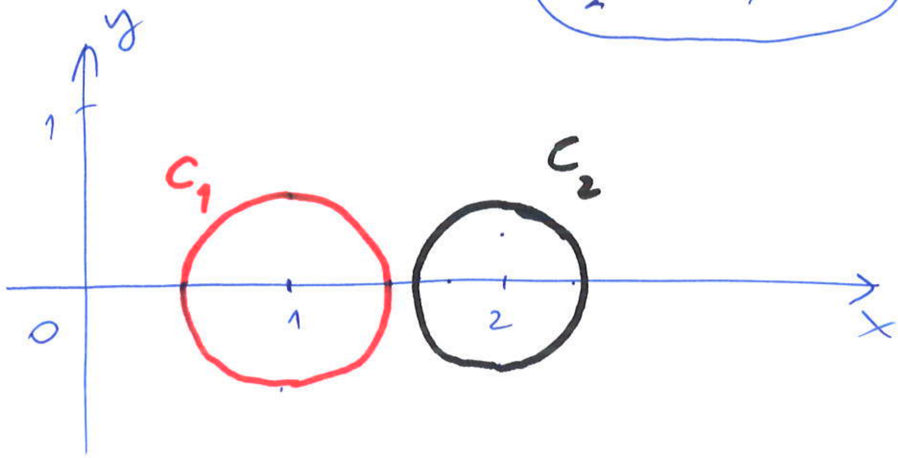
$(x-1)^2+y^2=\frac{1}{4}$

$S_1(1,0)$; $r_1=\frac{1}{2}=0,5$

$(x-2)^2+y^2=\frac{1}{8}$

$S_2(2,0)$; $r_2=\sqrt{\frac{1}{8}}=\frac{1}{2\sqrt{2}}=\frac{\sqrt{2}}{4}$

$r_2 \approx 0,354$



2.4.

$$D_n = \{(x,y) \in \mathbb{R}\times\mathbb{R},\ (x-n)^2 + y^2 \leq 2^{-n-1}$$

$$D_1 = \{(x,y) \in \mathbb{R}\times\mathbb{R},\ (x-1)^2 + y^2 \leq 2^{-2} = \frac{1}{4}$$

$$R_1^2 = \frac{1}{4} \qquad R_1 = \frac{1}{2}$$

$$R_2^2 = 2^{-2-1} = \frac{1}{2^3} \qquad R_2 = \sqrt{\frac{1}{2^3}}$$

$$S(D_1) = \pi \cdot R_1^2 = \pi \cdot \frac{1}{4} ; \qquad S_1(1,0)$$

$$S(D_2) = \pi \cdot R_2^2 = \pi \cdot \frac{1}{8} \qquad S_2(2,0)$$

$$S(D_3) = \pi \cdot R_3^2 = \pi \cdot \frac{1}{16} \qquad S_3(3,0)$$

$$q = \frac{1}{2} = \frac{S_{n+1}}{S_n}$$

$$S_\infty = \frac{S_1}{1-q} = \frac{\pi \cdot \frac{1}{4}}{1-\frac{1}{2}} = \frac{\pi \frac{1}{4}}{\frac{1}{2}} = \frac{\pi}{2}$$

$$S_\infty = \frac{\pi}{2}$$

$$D_1 \cup D_2 \cup D_3 \cup \ldots \cup D_n \cup \ldots = S_\infty$$

Naloga 3 je izbirna. Izbirate med nalogama 3 in 4 . Izbiro zaznamujte na naslovnici izpitne pole.

3.Realno zaporedje je podano rekurzivno

$$a_{n+2} = 2a_{n+1} - a_n\,; \quad a_1 = 2 \quad in \quad a_2 = 3$$

3.1.S popolno indukcijo dokažite, da za poljubno naravno število **n** velja

$$a_n = n + 1$$

3.2.Izračunajte vsoto vrste $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$

3.3.Izračunajte limite

$$\lim_{n\to\infty} \sqrt{\frac{2a_n}{a_n + 1}}$$

$$\lim_{x\to\infty} \frac{sinx}{a_n x} \quad , \text{ kjer je } \quad x\neq 0$$

$$\lim_{x\to 0} \frac{sinx}{a_n x} \quad , \text{ kjer je } \quad n\in \mathbb{N}$$

3.1. rekurzivno zaporedje

$$\left.\begin{aligned} a_{n+2} &= 2a_{n+1} - a_n \\ a_1 &= 2 \\ a_2 &= 3 \end{aligned}\right\} \Leftrightarrow a_n = n+1$$

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Dokaz z indukcijo:

1. $n=3$:

$$a_3 = 2a_2 - a_1 = \\ = 2\cdot 3 - 2 = \\ = 6-2 \\ = \underline{\underline{4}}$$

$$a_3 = 3+1 \\ = \underline{\underline{4}}$$

2. $m+1 \longrightarrow m+2$:

Dokazujemo:

$$a_{m+2} \overset{?}{=} 2a_{m+1} - a_n = m+3$$

$$a_{n+2} = 2a_{n+1} - a_n$$

Indukcijski predpostavki:

$a_n = n+1$

$a_{n+1} = n+2$

$$= 2(n+2) - (n+1) \\ = 2\cdot n + 4 - n - 1 \\ = \underline{\underline{n+3}}$$

3.2. $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3^n} + \sum_{n=1}^{\infty} \frac{2^n}{3^n} =$

$= \sum_{n=1}^{\infty} \frac{1}{3^n} + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{2} + 2 = \frac{5}{2}$

$$S_\infty = \frac{a_1}{1-q}$$

$$S_{\infty_1} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$$S_{\infty_2} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2$$

3.3. $\lim_{n\to\infty} \sqrt{\frac{2a_n}{a_{n+1}}} = \lim_{n\to\infty} \sqrt{\frac{2(n+1)}{(n+1)+1}} =$

$= \lim_{n\to\infty} \sqrt{\frac{2n+2}{n+2}} = \lim_{n\to\infty} \sqrt{\frac{2+\frac{2}{n}}{1+\frac{2}{n}}} = \sqrt{2}$

$$\lim_{x\to\infty} \frac{\sin x}{a_n \cdot \cancel{x}} = \lim_{x\to\infty} \frac{\frac{\sin x}{a_n}}{x} = \frac{K}{\infty} = 0$$

$$\lim_{x\to 0} \frac{\sin x}{a_n x} = \frac{1}{a_n} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{a_n} \cdot 1 = \frac{1}{a_n}$$

Naloga 4 je izbirna. Izbirate med nalogama 3 in 4. Izbiro zaznamujte na naslovnici izpitne pole.

4.V predavalnici je 40 stolov, ki so razdeljeni v 5 vrst tako da je v vsaki vrsti enako število stolov. Na stole se naključno posede 8 študentov matematike: Maja, Eva, Ela, Jan, Tim, Nik, Luka in France.

4.1.Izračunajte verjetnost dogodkov:

A- prva vrsta ostane prazna

B- v prvi vrsti so zasedeni natanko 3 stoli

C- vsi študenti so se posedli v isto vrsto

Maja, Eva, Ela, Jan, Tim, Nik, Luka in France ob popoldnevih igrajo družabne igre. Vsak natanko enkrat vrže pošteno igralno kocko.

4.2.Izračunajte verjetnost dogodkov:

D- nihče ne vrže šestico

E- natanko dva vržeta šestico

F- vsaj dva vržeta šestico

H- šestico vržeta samo Maja in France.

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Rešitve:

4.1.
$$P(A) = \frac{m}{n} = \frac{\binom{8}{0}\cdot\binom{32}{8}}{\binom{40}{8}} = \frac{1\cdot 10518300}{76904685} = 0{,}1368 = 13{,}68\%$$

$$P(B) = \frac{m}{n} = \frac{\binom{8}{3}\cdot\binom{32}{5}}{\binom{40}{8}} = \frac{56\cdot 201376}{76904685} = 0{,}1466 = 14{,}66\%$$

$$P(C) = \frac{\binom{8}{8}}{\binom{40}{8}}\cdot 5 = \frac{5}{76904685} = 6{,}502\cdot 10^{-8}$$

4.2.
$$P(D) = \frac{m}{n} = \left(\frac{5}{6}\right)^8 = 0{,}2326 = 23{,}26\%$$

$$P(E) = \binom{8}{2}\cdot\left(\frac{1}{6}\right)^2\cdot\left(\frac{5}{6}\right)^6 = 0{,}2605 = 26{,}05\%$$

$$P\left(\begin{matrix}\text{vsaj}\\\text{dva}\end{matrix}\right) = 1 - P\left(\begin{matrix}\text{niti}\\\text{eden}\end{matrix}\right) - P\left(\begin{matrix}\text{Natanko}\\\text{eden}\end{matrix}\right)$$

$$= 1 - \binom{8}{0}\left(\frac{1}{6}\right)^0\left(\cdot\frac{5}{6}\right)^8 - \binom{8}{1}\left(\frac{1}{6}\right)^1\cdot\left(\frac{5}{6}\right)^7$$

$$P(F) = 0{,}3953 = 39{,}53\%$$

$$P(H) = \frac{1}{6}\cdot\frac{1}{6}\cdot\left(\frac{5}{6}\right)^6 = 9{,}303\cdot 10^{-3}$$